On the Differential Equation $\Delta u + k^2 u = 0$.

By Maxime Bôcher.

It is well known that any solution of the differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0, \tag{1}$$

or as I will say for the sake of brevity any u-function, yields when multiplied by the factor $e^{\pm kz}$ a Newtonian potential function, that is a solution of the equation :

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \tag{2}$$

It has not, however, as far as I have been able to ascertain, been noticed that this fact can be made use of to deduce a considerable number of the fundamental properties of u-functions of two variables from well known properties of the Newtonian potential function. It is my purpose in the following paper to show in some detail how this can be done.

It is true that the method here suggested has only a limited range of application, and that from the point of view of the purist (I use the term in no invidious sense), the processes employed by H. Weber* and Pockels,† which consist in generalizing the methods and formulæ of the theory of the *two* dimensional potential, are vastly to be preferred. The course pursued in the present article, however, has the advantage of arriving at many of the most important results with very little labor when once the properties of the Newtonian potential function are premised.‡

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* "Ueber die Integration der partiellen Differentialgleichung : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0$." Mathematische Annalen, Vol. I.

† "Ueber die partielle Differentialgleichung $\Delta u + k^2 u = 0$." Teubner, Leipzig, 1891.

‡ In the present paper I shall confine my attention to u-functions with two independent variables. u-functions with n independent variables, i. e. solutions of the equation

$$\sum_{i=1}^{i=n} \frac{\partial^2 u}{\partial x_i^2} + k^2 u = 0,$$

may, however, be treated in precisely the same way. Their properties would then be made to depend upon the properties of the potential in space of $n + 1$ dimensions.

Instead of dealing with either of the Newtonian potential functions $e^{\pm kz} \cdot u(x, y)$, it will usually be more convenient to take as our potential a combination of the two, namely

$$\cosh kz \cdot u(x, y).$$

One advantage of this modification is that when k has the pure imaginary value $i\varkappa$ this potential is still real, having the form

$$\cos \varkappa z \cdot u(x, y).$$

The u-functions whose k is real have in many respects different properties from those whose k is pure imaginary. One of the most fundamental of these differences is the following:

A u-function with pure imaginary k cannot vanish at all points of the boundary of a region which lies in a portion of the x, y plane where the function is finite, continuous and single valued. When, however, k is real, u-functions do exist which vanish along the boundaries of such regions.*

In order to establish the first part of this proposition we have merely to notice that any vanishing line of a u-function in the x, y plane gives a cylindrical surface whose elements are parallel to the axis of z, at every point of which the potential $\cos \varkappa z \cdot u(x, y)$ vanishes. This potential, however, vanishes also on an infinite number of planes parallel to the plane x, y. Accordingly if there did exist a region of the nature above described on whose boundary the u-function vanished, the potential function would vanish on the boundary of an infinite number of finite solids cut out by the cylindrical surface just mentioned and by planes perpendicular to its elements. This, however, we know from the theory of the potential to be impossible.

On the other hand when k is real the above reasoning does not apply; for the potential $\cosh kz \cdot u(x, y)$ vanishes only on the cylindrical surfaces erected on the vanishing curves of the u-function, and space is not cut up into finite regions on whose boundaries the potential vanishes. That when k is real we actually do have such exceptional regions† in the x, y plane on whose boundaries the u-function vanishes is seen by numerous familiar examples.

Another proposition which we can obtain with ease is that, whether k be real or imaginary, *no point at which a u-function vanishes can be isolated, but every*

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* Pockels, p. 189 ; pp. 33–186.

† "Ausgezeichnete Bereiche"; cf. Pockels, p. 222.

such point must lie on a curve at all of whose points the u-function vanishes.[*]   For
if such an isolated point existed, the potential $\cosh kz . u(x, y)$ would vanish
along an isolated line, whereas we know that it must vanish along a whole
surface.

Now it is equally true that every equipotential surface $V =$ constant (not
merely those where $V = 0$) must be a real surface and not an isolated point or
line.   We cannot, however, conclude from this that the points where a u-function
has a given value must completely fill a line, for the surfaces along which
$\cosh kz . u(x, y)$ is constant are not cylindrical surfaces except when this constant
has the value zero.   The sections of these equipotential surfaces with the plane
x, y give the lines along which the u-function has constant values.   It is, how-
ever, perfectly conceivable that some of these equipotential surfaces should cut
the plane x, y in one or more isolated points.   That this really does occur is
seen most readily by considering the u-function $J_0(k\sqrt{x^2 + y^2})$ where J_0 denotes
a Bessels function of the first kind of order zero.

It should be noticed that isolated points of this kind, at which of course the
u-function reaches a maximum or a minimum, may occur not merely when k is
real but also when it is pure imaginary.   There is, however, an essential distinc-
tion between the two cases which is not explicitly stated by Weber or Pockels,[†]
and which may be stated as follows:

*Within a region where a u-function is finite, continuous and single valued there
may exist points where it reaches a maximum or a minimum.   If, however, k is real,
the* ABSOLUTE VALUE *of u must have a maximum at these points; if k is pure imaginary,
a minimum.*

To prove this proposition we may assume that the u-function has a positive
value at the point in question, for if it had not we could get a u-function which
would be positive there by multiplying by -1.   Then we merely have to show
that if k is real, u cannot be a minimum, and that if k is pure imaginary it cannot
be a maximum.   This, however, follows at once from the fact that k being real
and u a positive minimum, the potential $\cosh kz . u(x, y)$ would itself be a
minimum at the point in question; while if $k = i\varkappa$ were pure imaginary and u a
positive maximum, the potential $\cos \varkappa z . u(x, y)$ would be a maximum.

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[*] Pockels, p. 218.

[†] It is, however, implicitly contained in the last paragraph on page 10 of Weber's article quoted
above.   I find, however, no indication of the possibility of extending the proposition to the case where
u is a maximum or a minimum along a curve.

For the same reasons the above proposition holds true when for maximum or minimum *points* of the u-function we substitute maximum or minimum *lines*. The value of the u-function on such lines cannot, moreover, be zero any more than it can at maximum or minimum points. This also follows easily from the theory of the Newtonian potential function.*

Another proposition, whose truth we see at a glance, is the one established by Pockels (p. 226), which says that *two vanishing lines of a u-function cut each other, if at all, orthogonally, except when n vanishing lines pass through the same point, in which case they make equal angles $\frac{\pi}{n}$ with one another.* This follows from the corresponding proposition concerning the Newtonian potential, since the vanishing surfaces of the potential $\cosh kz \cdot u(x, y)$ are merely the cylinders erected on the vanishing lines of the u-function as base. Here again, however, we can draw no inference concerning the lines along which a u-function has a constant value other than zero.

The theory of the Newtonian potential tells us that a potential V can be developed about any non-singular point into a series proceeding according to ascending spherical harmonics, this development holding within a sphere described about the point as centre and passing through the nearest singular point of the potential function. If we apply this proposition to the special class of potentials we are here interested in we get the following development:

$$\cosh kz \cdot u(x, y) = \Phi_0 + r\Phi_1 + r^2\Phi_2 + \ldots.$$

where the surface spherical harmonic Φ_n may, by using polar coordinates, be written in the form:

$$\Phi_n =$$
$$a_{n,0}P_n(\cos\vartheta) + a_{n,1}P_n^1(\cos\vartheta) \cdot \cos\phi + a_{n,2}P_n^2(\cos\vartheta) \cdot \cos 2\phi + \ldots + a_{n,n}P_n^n(\cos\vartheta) \cdot \cos n\phi$$
$$+ b_{n,1}P_n^1(\cos\vartheta) \cdot \sin\phi + b_{n,2}P_n^2(\cos\vartheta) \cdot \sin 2\phi + \ldots + b_{n,n}P_n^n(\cos\vartheta) \cdot \sin n\phi.$$

We will suppose the origin of the system of polar coordinates r, ϑ, ϕ to be in the x, y plane, and the axis of the system to be perpendicular to this plane. If now in the development we set $z = 0$ (i.e. $\vartheta = \frac{\pi}{2}$) we get:

$$u = X_0 + rX_1 + r^2X_2 + \ldots. \tag{3}$$

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*It may be interesting to note that at infinity just the opposite of this is true, inasmuch as there a u-function may have zero as a maximum or minimum value, and that this is the only case where a u-function can fail to have a (real) singularity at infinity.

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where

$$X_n = A_{n,0} + A_{n,1} \cos \phi + A_{n,2} \cos 2\phi + \ldots + A_{n,n} \cos n\phi$$
$$+ B_{n,1} \sin \phi + B_{n,2} \sin 2\phi + \ldots + B_{n,n} \sin n\phi.*$$

We thus get the proposition:

In the neighborhood of any non-singular point a u-function may be developed in a series of the form (3), *and this series will converge within a circle described about the point in question as centre and passing through the nearest singular point of the u-function.*

If we compare the development (3) with the development

$$u = \sum_{n=0}^{n=\infty} J_n(kr) \cdot (A_n \cos n\phi + B_n \sin n\phi) \tag{4}$$

(cf. Pockels, p. 226) we see that each term of (4) is a *u*-function while the terms of (3) are not. This makes (4), as Pockels says, the natural generalization of the development

$$V(x, y) = \sum_{n=0}^{n=\infty} r^n \cdot (A_n \cos n\phi + B_n \sin n\phi) \tag{5}$$

of a two dimensional potential. The region of convergency of the series (4) has not, however, been investigated, while, as above shown, the series (3) has a circle of convergency exactly like that of (5).

A number of other propositions in the theory of the Newtonian potential yield in the same way simple properties of the two dimensional *u*-function.†

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* These functions X_n are considerably simplified, owing to the fact that all of their coefficients $A_{n,m}$ and $B_{n,m}$ vanish in which $n - m$ is an odd number, since the same is true of $P_n^m(0)$ which enters as a factor into them.

† On the other hand, many simple properties of the Newtonian potential yield propositions concerning the *u*-function which are so complicated as to be of comparatively little interest. Thus the proposition that the average value of a potential on the surface of a sphere is equal to the value of the potential at the centre of the sphere gives the following:

If r_1 is the radius of a circle which lies in a region of the x, y plane where a *u*-function is finite, continuous and single valued, and if r denotes the (variable) distance from the centre of this circle, then the average value of

$$\frac{r_1 \cosh (k\sqrt{r_1^2 - r^2})}{2\sqrt{r_1^2 - r^2}} \cdot u(x, y)$$

within the circle is equal to the value of u at the centre of the circle.

By comparing this complicated proposition with the simple one to which it bears a certain resemblance (equation 66, Pockels, p. 217) we get the following rather simple definite integral formula, which, of course, has nothing to do with the theory of *u*-functions:

$$x = \int_0^x \cos z \cdot J_0(\sqrt{z^2 - x^2}) dz.$$

One of the more important propositions which can be proved at once in this way will be found on page 212 of Pockels' book.

Still another application of our method should be mentioned, although the limits of the present article forbid a detailed discussion of it.

An extended class of problems in the theory of the potential may be solved by the method of development in series which proceed according to trigonometric functions, spherical harmonics, or other similar functions. The same is true in the theory of the u-functions.* In the theory of the potential a simple theory has been found by Klein† connecting the great number of isolated problems which had previously been solved by development in series. The same theory may be at once applied to u-functions (of any number of variables) by means of the method of the present article. I expect on another occasion to return to this question.

In conclusion we must note that the method of the present article is applicable not merely to the equation $\Delta u + k^2 u = 0$, but also to a number of other partial differential equations, chief among these being the equation for surface spherical harmonics. We could get by means of our method a series of propositions concerning surface spherical harmonics closely resembling those deduced in the present article for u-functions; and in one respect we should be even freer in the application of our method, as we should not be hampered by the presence of an essential singular point, such as we have at infinity in the case of the potential $\cosh kz \cdot u(x, y)$. The method employed by Prof. Klein in his lectures to prove that complete surface spherical harmonics must be of integral order‡ comes under the method of the present article.

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* See Pockels, pp. 326–335.

† See his note in the Göttinger Nachrichten for March, 1890; and also my essay: "Ueber die Reihenentwickelungen der Potentialtheorie," Göttingen, 1891.

‡ See Pockels, p. 106.